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A simple numerical model for small-scale meteorological process in the tropics

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Abstract: A simple numerical model is proposed enabling students to study small scale meteorological processes in the tropical atmosphere. These processes include heating and cooling of the air at the Earth's surface, and the effect of friction on the surface wind. Simplified equations represent the water in the atmosphere in vapor or liquid form as cloud. Precipitation is not included and Coriolis is omitted.

1. Introduction

This is a simple model in two space dimensions designed to show students how numerical weather prediction models are constructed [1]. It represents the behavior of air and the formation of clouds over level terrain in a tropical climate.

The independent variables are the horizontal dimension *x*, the vertical dimension *z*, and the time dimension *t*. The model is designed for systems of the order 10 km across. Accordingly, the horizontal length of the domain is 100 km, and the grid resolution is 1 km. In order to remove the need for lateral boundary conditions the right hand side of the domain is identified with the left hand side. This gives the domain the topological form of a cylinder. The vertical height is 16.2 km from sea level to the tropopause, divided into 24 layers increasing in thickness from 100 m at the surface to 1,250 m at the top. A model run will be 60 min with outputs every 5 min.

Some preliminary results on an earlier version of this model have shown that the model can give realistic representations of rising and falling air due to heating and cooling of the air at the surface. In the rising air cloud is formed, and falling air over a cooled surface produces waves of fog spreading out from the cooled area. Wind over a rough area of the surface was found to rise and produce waves in a manner similar to that of air blowing over a hill. It is hoped to publish results from this new model and extensions of it in future papers [2].

2. Model Equations

The model equations are derived from the fundamental system of partial differential equations of computational fluid dynamics in Euler form representing the variation of the fluid properties with time at fixed points in space [3]. Table 1 gives the notation for the quantities in the model equations.

The molecular viscosity terms are omitted. The only body forces are *friction* at the Earth's surface in the horizontal momentum equation, and *gravity* in the vertical momentum equation. The Coriolis force is omitted because the model is for small-scale processes near the equator. Heating and cooling of the air can occur at the Earth's surface to simulate the effects of radiation.

Moisture in the air is assumed to consist of water vapor and condensed cloud water. The effect of moisture on the temperature of the air is through the latent heat of condensation and evaporation of water vapor in saturated air. The effects of moisture on the other thermodynamic properties of the air are neglected. No distinction is made between liquid water and ice. The equations governing the condensation and evaporation of water above the freezing point are also used below the freezing point.

These simplifications will let students see the physical processes clearly without changing the qualitative features of the cloud processes represented.

Table 1. Notation for Model Constants and Variables.

Symbol	Value, Units	Description
x	m	Horizontal distance
z	m	Vertical height
t	s	Time
g	9.8 m/s^2	Acceleration of gravity
R	287 J/kg K	Gas constant for air
c_V	718 J/kg K	Specific heat of air at constant volume
L	2.5×10^6 J/kg	Latent heat of condensation of water
ρ	kg/m^3	Air density
u	m/s	Horizontal velocity
w	m/s	Vertical velocity
p	Pa	Pressure
T	K	Temperature
ρ_W	kg/m^3	Total water density (vapor plus liquid)
ρ_S	kg/m^3	Saturated water vapor density
m_C	kg/m^3	Condensed cloud water per unit volume
z_0	m	Roughness length of the surface
F_D		Drag factor at the surface
q_S	W/m^2	Surface heating rate per unit area
γ	K/m	Initial temperature lapse rate
δ	K	Initial dew point depression

2.1 The density equation

The air density forecast equation, derived from the mass continuity equation, is:

$$\frac{\partial \rho}{\partial t} = -u\frac{\partial \rho}{\partial x} - w\frac{\partial \rho}{\partial z} - \rho\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right). \qquad (1)$$

2.2 The wind equation

The wind forecast equation, derived from the horizontal momentum equation, is:

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - w\frac{\partial u}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial x} - F_D u|u|. \quad (2)$$

The last term is horizontal friction, which is applied only in the layer of air at the Earth's surface. Let z_1 be the height at which the wind in the surface layer is modeled. Then the drag coefficient C_D is given [4] by:

$$C_D = \left[\frac{k}{\ln(z_1/z_0)}\right]^2,$$

where k = 0.40 (von Karman's constant) and z_0 is the roughness length of the surface. This gives the drag factor

$$F_D = \frac{C_D}{\Delta z},$$

where Δz is the thickness of the surface layer.

2.3 The vertical velocity equation

The vertical velocity forecast equation, derived from the vertical momentum equation, is:

$$\frac{\partial w}{\partial t} = -u\frac{\partial w}{\partial x} - w\frac{\partial w}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} - g. \quad (3)$$

2.4 The moisture equations

The total water density forecast equation (vapor plus liquid), derived from the continuity equation, is:

$$\frac{\partial \rho_w}{\partial t} = -u\frac{\partial \rho_w}{\partial x} - w\frac{\partial \rho_w}{\partial z} - \rho_w\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right). \quad (4)$$

The temperature equation used depends on whether or not the air is saturated. A simple approximate equation for the saturated vapor density ρ_s of water as a function of temperature is obtained by integrating the Clausius-Clapeyron equation assuming that the latent heat of condensation of water vapor is constant and water vapor is an ideal gas [5]:

$$\rho_s = \frac{A}{T}e^{-B/T}, \quad (5)$$

where $A = 549 \times 10^6$ kg K/m^3, and $B = 5.42 \times 10^3$ K. If the temperature changes, the saturated vapor density changes in accordance with the equation

$$\frac{d\rho_s}{dt} = \frac{A(B-T)}{T^3}e^{-B/T}\frac{dT}{dt}. \quad (6)$$

If $\rho_w < \rho_s$ the air is unsaturated; all the moisture in the air is water vapor and the condensed cloud water per unit volume m_c is zero. If $\rho_w \geq \rho_s$, then the air is saturated, the water vapor density is equal to ρ_s, and the condensed cloud water per unit volume m_c is given by the diagnostic equation

$$m_c = \rho_w - \rho_s.$$

2.5 The temperature equations

Unsaturated air

If the air is unsaturated, then the temperature forecast equation, derived from the thermodynamic energy equation, is:

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} - w\frac{\partial T}{\partial z} - \frac{RT}{c_v}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + \frac{q_s}{c_v\rho\Delta z}. \quad (7)$$

The surface heating term q_s is applied only in the layer of air at the Earth's surface.

Saturated air

In a Lagrangian parcel of saturated air the rate of change of the saturated vapor density is given by equation (6). The rate of change of temperature is then

$$\frac{dT}{dt} = -\frac{RT}{c_v}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + \frac{q_s}{c_v\rho\Delta z} - \frac{L}{c_v\rho}\frac{d\rho_s}{dt}.$$

Eliminating $d\rho_s/dt$ gives the temperature forecast equation in saturated air:

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} - w\frac{\partial T}{\partial z} - \frac{\frac{RT}{c_v}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + \frac{q_s}{c_v\rho\Delta z}}{1 + \frac{LA(B-T)e^{-B/T}}{c_v\rho T^3}}. \quad (8)$$

2.6 The pressure equation

The diagnostic pressure equation is the ideal gas equation

$$p = \rho RT. \quad (9)$$

3. Numerical Approximations to the Model Equations

3.1 The domain

The domain of the model is divided into a 100 × 24 array of cells. The cells are labeled horizontally from i = 1 to i = 100, and vertically from k = 1 to k = 24. The horizontal width of each cell is Δx = 1 km.

For practical computing, two additional vertical columns of cells are added on each side of the domain, one labelled i = 0 and the other labelled i = 101. The values of the model variables in the i = 0 cell are equal to the values of the corresponding variables in the i = 100 cell, and the values of the model variables in the i = 101 cell are equal to the values of the corresponding variables in the i = 1 cell. This makes the domain a vertical cylinder without any lateral boundaries.

3.2 Stretched vertical grid

A stretched vertical grid is used to give a fine resolution in the planetary boundary layer and a coarse resolution at the top of the troposphere. A dimensionless vertical coordinate s is used in the numerical approximations in accordance with the transformation.

$$\begin{aligned} s &= \sqrt{2.25 + 0.04z} - 1.5 \\ z &= 75s + 25s^2, \end{aligned}$$

where z is in meters. The grid is shown in Fig. 1. The 24 levels go up to a height 16,200 m, which is assumed to be the tropopause. The thickness of the layer at the Earth's surface (k = 1, s = 0 to s = 1) is 100 m, and the thickness of the top layer (k = 24, s = 23 to s = 24) is 1,250 m.

3.3 The staggered grid

The model variables ρ, u, w, T, ρ_w, and m_c are evaluated at points on a staggered grid as shown in Fig. 2. In the (i,k)-cell the air density $\rho_{i,k}$, temperature $T_{i,k}$, total water density $\rho_{w(i,k)}$ and condensed cloud water amount per unit volume $m_{c(i,k)}$, are in the center of the cell, at height z defined by $s = k - 0.5$. The horizontal velocity $u_{i,k}$ is on the left side of the cell, and the vertical velocity $w_{i,k}$ is on the bottom of the cell.

3.4 Steps in time

The progress of the model variables in time is calculated by a second order *Runge-Kutta* method using a predictor step followed by a corrector step, each over the same time interval of length Δt. The model variables have labels n = 1, 2 and 3, which are used for the present value, the predicted value, and the corrected value, respectively. Let $Y^1_{i,k}$ represent the present value of one of the model variables in the (i, k)-cell. Then the predicted value $Y^2_{i,k}$ the variable is calculated by a forward Euler step

$$Y^2_{i,k} = Y^1_{i,k} + F^1 \Delta t,$$

where F^1 represents the function of the model variables and the approximations to the space derivatives on the right hand side of the forecast equation for $Y^2_{i,k}$ with n = 1. The corrected value $Y^3_{i,k}$ is calculated by the equation

$$Y^3_{i,k} = Y^1_{i,k} + [(1-a)F^1 + aF^2]\Delta t$$

where F^2 represents the same function as F^1 calculated from the predicted values $Y^2_{i,k}$ with n = 2, and a is an adjustable correction factor. If a = 0 the method gives a simple Euler step; if a = 0.5 the method gives a Heun step;
and if a = 1 the method gives a Matsuno step. Other values of the correction factor may be used. For example a = 0.75 gives a method between the Heun and the Matsuno methods.

3.5 Approximations in space

The difference approximations given below are used in the cells of the domain where i = 1 to 100 and k = 1 to 24. In the row of cells at the Earth's surface (k = 1) and at the top of the troposphere (k = 24) one-sided approximations to derivatives with respect to the vertical coordinate are used. In the other rows of cells (k = 2 to k = 23) central approximations to derivatives with respect to the vertical coordinate are used. In the horizontal direction Δx = 1000 m. The vertical difference approximations are calculated in the the computational domain assuming that $s = k - 1$ at the bottom of the k-th cell, and $s = k - 0.5$ in the middle of the k-th cell. The formulas below contain numerical values using the physical quantities *in base SI units* as shown in Table 1.

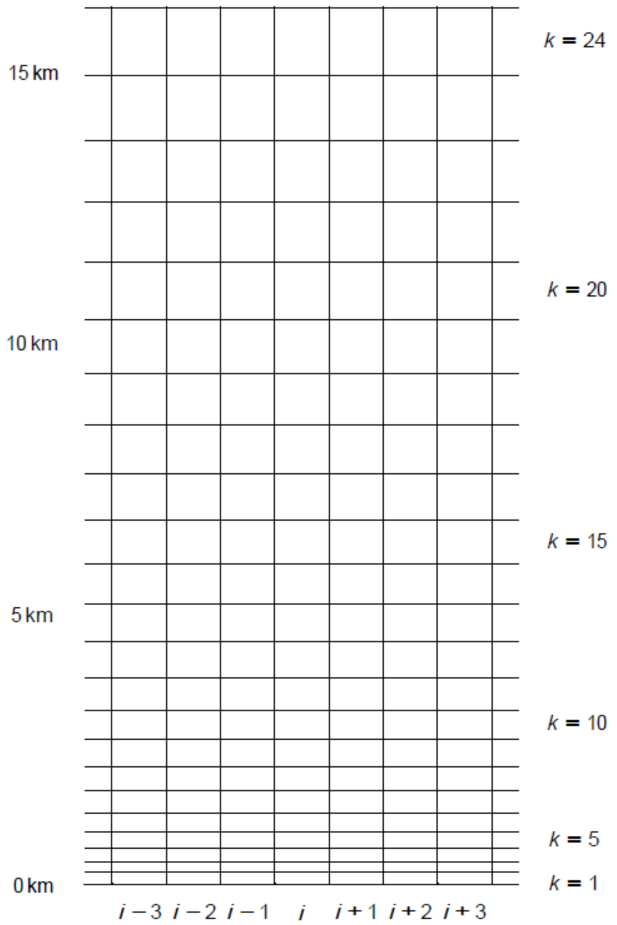

Figure 1. The stretched vertical grid showing the heights and labels of the cells. All the cells have horizontal width 1 km.

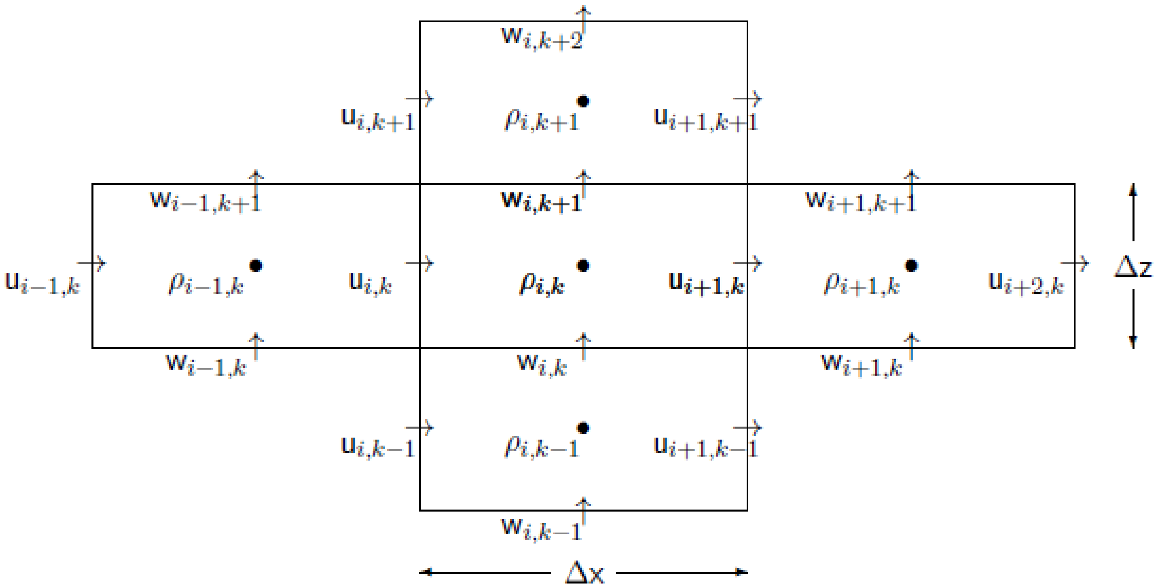

Figure 2. The cell (i,k) and its neighbors showing the locations of the model variables. The temperatures T, total water densities ρ_W, and cloud water amounts m_C per unit volume are located at the same points as the densities ρ in the middle of the cells.

The density equation

The numerical approximations for the terms on the right side of the density equation (1) in the (i,k)-cell are as follows:

$$u\frac{\partial \rho}{\partial x} \approx \frac{(u_{i,k}+u_{i+1,k})(\rho_{i+1,k}-\rho_{i-1,k})}{4000}.$$

For k = 1

$$w\frac{\partial \rho}{\partial z} \approx \frac{w_{i,2}(-3\rho_{i,1}+4\rho_{i,2}-\rho_{i,3})}{400},$$

for k = 2 to k = 23

$$w\frac{\partial \rho}{\partial z} \approx \frac{(w_{i,k}+w_{i,k+1})(\rho_{i,k+1}-\rho_{i,k-1})}{200(k+1)},$$

and for k = 24

$$w\frac{\partial \rho}{\partial z} \approx \frac{w_{i,24}(\rho_{i,22}-4\rho_{i,23}+3\rho_{i,24})}{5000}.$$

For the last term in equation (1) we have:

$$\rho\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right) \approx \rho_{i,k}\left(\frac{u_{i+1,k}-u_{i,k}}{1000}+\frac{w_{i,k+1}-w_{i,k}}{50(k+1)}\right).$$

The wind equation

The numerical approximations for the terms on the right side of the wind equation (2) are as follows:

$$u\frac{\partial u}{\partial x} \approx \frac{u_{i,k}(u_{i+1,k}-u_{i-1,k})}{2000}.$$

For k = 1

$$w\frac{\partial u}{\partial z} \approx \frac{(w_{i-1,2}+w_{i,2})(-3u_{i,1}+4u_{i,2}-u_{i,3})}{800},$$

for k = 2 to k = 23

$$w\frac{\partial u}{\partial z} \approx \frac{(w_{i-1,k}+w_{i,k}+w_{i-1,k+1}+w_{i,k+1})(u_{i,k+1}-u_{i,k-1})}{400(k+1)},$$

and for k = 24

$$w\frac{\partial u}{\partial z} \approx \frac{(w_{i-1,24}+w_{i,24})(u_{i,22}-4u_{i,23}+3u_{i,24})}{10000}.$$

The ideal gas equation (9) is used to eliminate pressure:

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = R\left(\frac{T}{\rho}\frac{\partial \rho}{\partial x}+\frac{\partial T}{\partial x}\right),$$

so that

$$\frac{1}{\rho}\frac{\partial p}{\partial x} \approx 0.287\left(\frac{(T_{i-1,k}+T_{i,k})(\rho_{i,k}-\rho_{i-1,k})}{(\rho_{i-1,k}+\rho_{i,k})} + (T_{i,k}-T_{i-1,k})\right).$$

If k = 1, then

$$F_D \approx \frac{0.0016}{[\ln(43.75/z_{0(i)})]^2},$$

since the mid point of the surface layer k = 1 in the computational domain, given by s = 0.5, is 43.75 m, and

$$F_D u|u| \approx F_{D(i)} u_{i,1}|u_{i,1}|.$$

If k > 1 this friction term is zero.

The vertical velocity equation

The numerical approximations for the terms on the right side of the vertical velocity equation (3) are as follows:

$$w\frac{\partial u}{\partial z} \approx \frac{(u_{i,k-1}+u_{i+1,k-1}+u_{i,k}+u_{i+1,k})(w_{i+1,k}-w_{i-1,k})}{8000}.$$

$$w\frac{\partial w}{\partial z} \approx \frac{w_{i,k}(w_{i,k+1}-w_{i,k-1})}{50(2k+1)}.$$

The ideal gas equation (9) is used to eliminate pressure:

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = R\left(\frac{T}{\rho}\frac{\partial \rho}{\partial z}+\frac{\partial T}{\partial z}\right),$$

so that

$$\frac{1}{\rho}\frac{\partial p}{\partial z} \approx \frac{11.48}{(2k+1)}\left(\frac{(T_{i,k-1}+T_{i,k})(\rho_{i,k}-\rho_{i,k-1})}{(\rho_{i,k-1}+\rho_{i,k})} + (T_{i,k}-T_{i,k-1})\right).$$

The moisture equation

The numerical approximations for the terms on the right side of the total water density equation (4) in the (i,k)-cell are as follows:

$$u\frac{\partial \rho_w}{\partial x} \approx \frac{(u_{i,k}+u_{i+1,k})(\rho_{w(i+1,k)}-\rho_{w(i-1,k)})}{4000}.$$

For k = 1

$$w\frac{\partial \rho_w}{\partial z} \approx \frac{w_{i,2}(-3\rho_{w(i,1)}+4\rho_{w(i,2)}-\rho_{w(i,3)})}{400},$$

for k = 2 to k = 23

$$w\frac{\partial \rho_w}{\partial z} \approx \frac{(w_{i,k}+w_{i,k+1})(\rho_{w(i,k+1)}-\rho_{w(i,k-1)})}{200(k+1)},$$

and for k = 24

$$w\frac{\partial \rho_w}{\partial z} \approx \frac{w_{i,24}(\rho_{w(i,22)}-4\rho_{w(i,23)}+3\rho_{w(i,24)})}{5000}.$$

For the last term in equation (4) we have:

$$\rho_w\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right) \approx \rho_{w(i,k)}\left(\frac{u_{i+1,k}-u_{i,k}}{1000}+\frac{w_{i,k+1}-w_{i,k}}{50(k+1)}\right).$$

The temperature equation

Before calculating the changes in temperature T, the total water density ρ_W is compared with the saturated vapor density ρ_S given by equation (5). If $\rho_W < \rho_S$, then the air is unsaturated, m_C is put equal to zero, and equation (7) is used. If $\rho_W \geq \rho_S$, then the air is saturated, m_C is put equal to $\rho_W - \rho_S$, and equation (8) is used.

The numerical approximations to the expressions on the right side of the temperature forecast equations (7) are as follows:

$$u\frac{\partial T}{\partial x} \approx \frac{(u_{i,k} + u_{i+1,k})(T_{i+1,k} - T_{i-1,k})}{4000}$$

For $k = 1$

$$w\frac{\partial T}{\partial z} \approx \frac{w_{i,2}(-3T_{i,1} + 4T_{i,2} - T_{i,3})}{400},$$

for $k = 2$ to $k = 23$

$$w\frac{\partial T}{\partial z} \approx \frac{(w_{i,k} + w_{i,k+1})(T_{i,k+1} - T_{i,k-1})}{200(k+1)},$$

and for $k = 24$

$$w\frac{\partial T}{\partial z} \approx \frac{w_{i,24}(T_{i,22} - 4T_{i,23} + 3T_{i,24})}{5000}.$$

$$\frac{RT}{c_v}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) \approx 0.4T_{i,k}\left(\frac{u_{i+1,k} - u_{i,k}}{1000} + \frac{w_{i,k+1} - w_{i,k}}{50(k+1)}\right)$$

If $k = 1$, then

$$\frac{q_s}{c_v \rho \Delta z} = \frac{q_{s(i)}}{71800\rho_{i,1}},$$

Otherwise

$$\frac{q_s}{c_v \rho \Delta z} = 0.$$

$$\frac{LA(B-T)e^{-B/T}}{c_v \rho T^3} \approx \left(\frac{1.912 \times 10^{12}(5.42 \times 10^3 - T_{i,k})}{\rho_{i,k} T_{i,k}^3}\right) \exp\left(\frac{-5.42 \times 10^3}{T_{i,k}}\right)$$

4. Running the Model

4.1 Initial conditions

The initial values of the model variables in each cell are functions of the height of the cell above the Earth's surface, but are constant along the horizontal rows of cells.

Temperatures

The initial temperatures T_k^0 in the domain are given by:

$$T^0(z) = 300 - \gamma z,$$

where $T^0(z)$ is in kelvins. The default value of the initial temperature lapse rate γ is 0.00675 K/m to approximate the annual mean upper air tempera- tures at Bangkok [6]. These temperatures are given by

$$T_k^0 = 300 - \gamma(31.25 - 50k - 25k^2).$$

Air densities

The initial density at height 43.75 m in the bottom row of cells, based on annual mean data for Bangkok [7], is

$$\rho_1^0 = 1.17\,\mathrm{kg\,m^{-3}}.$$

The initial upper air densities are calculated from this value assuming the hydrostatic condition using the formula

$$\rho_k^0 = \left(\frac{2.3428571T_{k-1}^0 - (2k+1)}{2.3428571T_k^0 + (2k+1)}\right)\rho_{k-1}^0,$$

which ensures that no false vertical velocities are produced by unbalanced vertical pressure forces in the initial state.

Horizontal velocities

The default initial horizontal velocities $u_{i,k}$ are all zero, but the user can set arbitrary values as a function of height.

Vertical velocities

The initial values of the vertical velocities $w_{i,k}$ are all zero.

Moisture

The initial values of the total water density are calculated on the assumption that the dewpoint depression below the initial air temperature is a constant δ at all heights z. The default value of the dewpoint depression is $\delta = 5$ K, but this may be reset by the user to experiment with different amounts of moisture in the air.

The initial upper air total water densities, given by equation (5), are then:

$$\rho_{w(k)}^0 = \frac{A}{(T_k^0 - \delta)} \exp\left(\frac{-B}{T_k^0 - \delta}\right),$$

and the initial amounts $m_{c(k)}^0$ of cloud water per unit volume are all zero.

4.2 Boundary conditions

The following boundary conditions are maintained throughout a model run:

Vertical velocities

The vertical velocities w are zero on the bottom and top boundaries of the domain where w is labeled k = 1 and 25.

Surface roughness lengths

The surface roughness lengths z_0 at the surface of the Earth are defined on the left sides of the surface cells (k = 1). The default values of z_0 are 0.001 m for a smooth surface along the horizontal length of the domain.

The roughness lengths z_0 can be given different values in different positions to simulate variations in the roughness of the terrain, e.g. 10 m for a very rough surface, such as a city with tall buildings.

Surface heating rates

The default initial surface heating rates $q_{s(i)}$ are zero everywhere. However, the user can reset the surface heating rates $q_{s(i)}$ to different values to simulate the effects of solar and terrestrial radiation.

5. Experiments with the Model

The following experiments will be tried to test the ability of the model to give reasonable representations of small-scale meteorological processes.

- A heated area in the middle of a cooled area.
- A cooled area in the middle of a heated area.
- Random heating and cooling at the surface in space and time across the domain.
- A rough area in the middle of a smooth surface.
- A smooth area in the middle of a rough surface.
- Random variation of surface roughness across the domain.
- A lake in a forest.
- A city heat island.
- Land and sea breezes.
- Cloud formation, or the absence of cloud, in air with high and low humidity.
- The effect of variable surface properties on the formation of clouds.
- The effect of vertical wind shear on cloud forms

6. Future Versions of the Model

Future versions of the model may contain the following features:

- Heating and cooling of the air by the interaction of shortwave and longwave radiation with condensed cloud water and the Earth's surface.
- Evaporation of moisture from the Earth's surface.
- Two horizontal dimensions.
- The use of fixed values of the model variables on the lateral boundaries with relaxation at the boundaries to prevent the reflection of advection waves.
- Variations in the height of the Earth's surface above sea level.

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